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Global Existence and Long Time Behavior for the Davey-Stewartson Systems

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1 Introduction

A large amount of work (cf. [1-19]) has been devoted to the study of genercalized of Davey-Stewartson systems:

$$\begin{cases} iu_t+\sigma u_{xx}+u_{yy}=\lambda|u|^2u+\mu uv_x \\ u_{xx}+\nu v_{yy}=(|u|^2)_x \end{cases} \tag{1.1}$$

Where u is the complex amplitude, a complex-valued function of $(t;x,y)\in R^+\times R^2$, $\mathcal{V}$ is the real mean velocity potential, a real-valued function of $(t;x,y)\in R^+\times R^2,\sigma,\lambda,\mu,\nu\in R$. The Davey-Stewartson systems were first derived by Davey and Stewartson [1] in the context of water waves, these systems model the evolution of weakly nonlinear water waves that travel predominantly in one direction, but in which the wave applitude is modulated slowly in two horizontal directions. The real parameters σ,λ,μ, and ν can assume both signs. Davey-Stewartson systems can be classified as elliptic-elliptic, elliptic-hyperbolic , hyperbolic-elliptic, and hyperbolic-hyperbolic according to the respective sign of $(\sigma,\nu):(+,+),(+,-),(-,+)$ and $(-,-)$ .System (1.1) were also derived by Djordjevic and Redehkopp [2], and Ablowitz and Haberman [3].Varions properties of solution for Davey-Stewartson systems have been investigated by many authors including Ghidaglia and Saut [4], Anker and Freeman[5], Ablowitz and

Fokas [6], M. Tsutsumi [7] Hayashi and Saut [8], Linares and Ponce [9] and thereferences therein.

In 1990, Ghidaglia and Saut in [4] studied the Cauchy problem of (1.1) and except for the case σ, $\nu < 0$ proved the solvability in the Sobolev spaces $H^1 = H^1(R^2)$. In the elliptic-hyperbolic case, *i.e.*,$\sigma > 0$ and $\nu < 0$ Tsutsumi in [7] obtained the $L^p(R^2)$ decay estimates of the solution of (1.1) $(2 < p < \infty)$. Ozawa in [10] preseuted the exact blow up solution of the Cauchy problem (1.1)., Ohta [11] and [12] discussed the existence and nonexistence of stable standing waves under certain conditions. In 1999, Guo and Wang [13] proved the global existence for the Cauchy problem (1.1) in $H^s(1 \leq s \leq 2, n = 2, 3)$. In 2001, they in [14] also extend this result to generalized (1.1) systems., Moreover, the existence of global attractors , global existence and blow up of solutions to a degenerate Davey-Stewartson equations and approximate inertial manifolds was also studied by Wang and Guo in [15], Li, Guo and Jiang in [16], and Guo, Li and Lin in [17], respectively .

In this paper, we shall first treat the case $\sigma > 0$ and $\nu > 0$ of (1.1), by using Besov space. Secoand we consider the existance of global attractor for (1.1)in R^2 and construct the approximate inertial manifolds to (1.1). Finally, the global existence and blow up of solution to degenerate Davey-Stewartson equations also have been established

2 The Cauchy problem for Davey-Stewartson systems

In this section , we shall treat the case $\sigma > 0$ and $\nu > 0$ and study the Cauchy problem of the following generalized Davey-Stewarstson systems,

$$\begin{cases} iu_t + \Delta u &= a|u|^\alpha u + b_1 u v_x \\ -\Delta v &= b_2(|u|^2)_{x1} \\ u(0,x) &= u_0(x) \end{cases} \tag{2.1}$$

Where $u(t,x)$ and $v(t,x)(x = (x_1 \cdots x_n))$ are complex and real valued functions of $(t,x) \in R^+ \times R^n$ respectively, Δ is the Laplace operator on R^n, and a, b_1 and b_2 are real constants . One can easily see that (2.1)is a generalized version of (1.1) is the case $\sigma > 0$ and $\nu > o$. let u, v be the solution of (2.1), It follows from the second equation in (2.1), that

$$v_{x_1} = -b_2 \mathcal{F}^{-1}(\frac{\xi_1^2}{|\xi|^2})\mathcal{F}|u|^2 \tag{2.2}$$

For brevity we denote

$$E(\psi) = \mathcal{F}^{-1}(\frac{\xi_1^2}{|\xi|^2})\mathcal{F}\psi \tag{2.3}$$

Combining (2,1) and (2,2), we have

$$\begin{cases} iu_t + \Delta u = a|u|^\alpha u - b_1 b_2 E(|u|^2)u \\ u(0,x) = u_0(x) \end{cases} \tag{2.4}$$

One can easily verify that (2,4)is essentially equivalent to (1.1) through the transformation (2.2). We shall study the local and global existence of solutions in $H^s(1 \leq s \leq 2)$ of problem (2.4) is two and three space dimensions.

1. We first state the main results, Let

$$\alpha_s(n) = \begin{cases} \infty, & s \geq \frac{n}{2}, \\ \frac{4}{n-2s} & 0 \leq s < \frac{n}{2}. \end{cases} \tag{2.5}$$

Theorem 2.1 Let $n = 2, 3$. $1 \leq s \leq 2$ and $u_0 \in H^s$. Suppose that $\alpha \in (1, \alpha_s(n)]$. Then there exists a unique solution u of (2.4) satisfying

$$u \in C_{loc}(0, T^*; H^s) \bigcap L_{loc}^{\gamma(r)}(o, T^*; H^{s,r})$$

for some $T^* \in (0, \infty)$, where $r \in [2, 2 + \alpha_1(n)]$ and $\frac{2}{\gamma(r)} = n(\frac{1}{2} - \frac{1}{r})$, Moreover, if $T^* < \infty$, then

$$\lim_{t \to T^*} sup||u(t)||_{H^s} = \infty$$

Theorem 2.2 let $n = 2, 1 \leq s \leq 2$, and $u_0 \in H^s$. Suppose that one of the following conditions holds:

(i) $a > 0$ *and* $2 < \alpha < \infty$;

(ii) $\alpha = 2$ *and* $a \leq max(0, b_1 b_2)$

(iii) $\alpha = 2$ *and* $b_1 b_2 \geq 0$, *and* $(b_1 b_2) - a)||u_0||_{L^2}^2 < 4$;

(iv) $\alpha = 2$ *and* $b_1 b_2 < 0$, *and* $-a||u_0||_{L^2}^2 < 4$;

(v) $1 \leq \alpha < 2$ *and* $b_1 b_2 ||u_0||_{L^2}^2 < 4$

Then (2.4) has a unique solution $u \in C_{loc}(0, \infty; H^s) \bigcap L_{loc}^{\gamma}(0, \infty; H^{s,r}) \bigcap C(0, \infty, H^1)$ for any $r \in [2, \infty)$and $\frac{2}{\gamma(r)} = 1 - \frac{2}{r}$.

Theorem 2.3 let $n = 3$, $1 \le s \le 2$, and $u_0 \in H^s$. Suppose that one of the following conditions holds:

$$\text{(i)} \quad a > 0, \quad 2 < \alpha < 4, \quad or$$

$$\text{(ii)} \quad \alpha = 2, \quad a > 0, \qquad and \ a \ge b_1 b_2$$

Then (2.4) has a unique solution $u \in C_{loc}(0,\infty; H^s) \cap L^{\gamma(r)}_{loc}(0,\infty; H^{s,r}) \cap C(0,\infty; H^1)$ for any $r \in [2,6)$ and $\frac{2}{\gamma(r)} = n(\frac{1}{2} - \frac{1}{r})$.

Throughout this paper, we will have occasion to use a variety of function spaces, Lebesque space $L^r = L^r(R^n)$; Bessel potential space $H^{s,r} = H^{s,r}(R^n)$, $H^s = H^{s,2}$; Riesz potential space $\dot{H}^{s,r} = \dot{H}^{s,r}(R^n), \dot{H}^s = \dot{H}^{s,2}$; Besov space $B^s_{r,q} = B^s_{r,q}(R^n), B^s_r = B^s_{r,2}$; and homogeneous Besov space $\dot{B}^s_{r,q} = \dot{B}^s_{r,q}(R^n)$, $\dot{B}^s_r = B^s_{r,2}$, The definitions of these spaces allow $1 < r, q < \infty, s \in R$. If $s > 0$, Then we have $B^s_r = L^r \cap \dot{B}^s_r$, $H^{s,r} = L^r \cap \dot{H}^s_r$ An equivalent definition of the norm on $\dot{B}^s_r$ is that

$$||u||_{\dot{B}^s_r} = \left(\int_0^\infty t^{-2(s-[s])} \sum_{|\alpha|=|s|} \sup_{|h|\le t} ||\triangle_h D^\alpha u||^2_{L^r} \frac{dt}{t} \right)^{\frac{1}{2}}, \tag{2.6}$$

where [s] denotes the largest integer less than or equal to s , $\triangle_h u(\cdot) = u(\cdot + h) - u(\cdot) = u_h - u$. For some additional basic results on Besov space , one can refer to [20]and [21].

In the following, C will stand for a constant , depending only on R^n, that can be different at different places . For any $r \in [1,\infty], r'$ denotes the duality number of $r, i.e., \frac{1}{r} + \frac{1}{r'} = 1$.

The main tools used in here are time-space$L^p - L^{p'}$ estimates for solution of linear Schrödinger equations in Lebesque-Besov spaces; these estimates are usually named generalized Strichartz inequalities. The method of the proof of main results is a contraction mapping argument. Les us recall that some estimates for linear Schrödinger equations in Lebesgue-Besov spaces have been established by Cazenave and Weissler in [12]

Proposition 2.4 Let $S(t) = e^{it\Delta}$. Let $s \in R$, $2 \leq r, \rho < 2 + \alpha_1(n)$, and let

$$\frac{2}{r(\cdot)} = n(\frac{1}{2} - \frac{1}{\cdot}) \tag{2.7}$$

(i) If $\varphi \in \dot{H}^s$, Then $S(\cdot)\varphi \in L^{\gamma(s)}(R, \dot{B}^s_r)$, and ther exists a constant $C > 0$ such that

$$||S(t)\varphi||_{L^{\gamma(r)}(R,\dot{B}^s_r)} \leq C||\varphi||_{\dot{H}^s} \tag{2.8}$$

for all $\varphi \in \dot{H}^s$

(ii) If $f \in L^{\gamma(r)'}(o, T; B^s_{r'})$, then $\int_0^t S(t-\tau)f(\tau)d\tau in L^{\gamma(\rho)}(o, T; \dot{B}^s_\rho)$, and there exist $C > o$ such that

$$||\int_0^t S(t-\tau)f(\tau)d\tau||_{L^{\gamma(\rho)}(o,T;\dot{B}^s_\rho)} \leq c||f||_{L^{\gamma(r)'}(o,T;\dot{B}^s_{r'})} \tag{2.9}$$

for all $f \in L^{\gamma(r)'}(o, T; \dot{B}^s_{r'})$, where $\frac{1}{\gamma(r)} + \frac{1}{\gamma(r)'} = 1$

2. Nonlinear estinates

Lemma 2.5 Let $1 \leq \lambda$, γ, $\sigma < \infty$, $\frac{1}{\lambda} = \frac{1}{\gamma} + \frac{1}{\rho}$,

(i) We have

$$||uv||_{H^{1,\lambda}} \leq C(||u||_{L^\rho}||v||_{H^{1,\gamma}} + ||u||_{H^{1,\rho}}||v||_{L^\gamma}) \tag{2.10}$$

for any $u \in H^{1,\rho}$ and $v \in H^{1,\gamma}$

(ii) Let $1 < s < 2$. Then we have

$$\begin{aligned} ||uv||_{B^s_\lambda} \quad &\leq \quad C(||u||_{L^\rho}||v||_{B^s_\gamma} + ||u||_{B^s_\rho}||v||_{L^\gamma} \\ &+ ||u||_{H^{1,\rho}}||v||_{B^{s-1}_\gamma} + ||u||_{B^{s-1}_\rho}||v||_{H^{1,\gamma}}) \end{aligned} \tag{2.11}$$

for all $u \in B^s_\rho$ and $v \in B^s_r$.

(iii) We have

$$||uv||_{H^{2,\lambda}} \leq C(||u||_{L^\rho}||v||_{H^{2,\gamma}} + ||u||_{H^{2,\rho}}||v||_{L^\gamma} + ||u||_{H^{1,\rho}}||v||_{H^{1,\gamma}}) \tag{2.12}$$

for any $u \in H^{2,\rho}$ and $V \in H^{2,\gamma}$.

Corollary 2.6. let $1 < s < 2$, We have

$$||uv||_{H^{1,4/3}} \leq C(||u||_{L^4}||v||_{H^1} + ||u||_{H^{1,4}}||\gamma||_{L^2}) \tag{2.13}$$

$$||uv||_{H^{2,4/3}} \leq C(||u||_{L^4}||v||_{H^2} + ||u||_{H^{2,4}}||v||_{L^2} + ||u||_{H^{1,4}}||v||_{H^1}) \tag{2.14}$$

$$||uv||_{B^s_{4/3}} \leq (C||u||_{L^4}||v||_{H^s} + ||v||_{L^2}||u||_{B^s_4} + ||u||_{H^{1,4}}||v||_{H^{s-1}} + ||v||_{H^1}||u||_{B^{s-1}_4})$$

Lemma 2.7 (Convexity Hölder Inequality)Assume that $1 < p_i, q_i \leq \infty$, $0 \leq \theta_i \leq 1, \sigma_i, \sigma \in R(i = 1, \cdots, N)$, $\sum_{i=1}^N \theta_i = 1$, $\sigma < \sum_{i=1}^N \theta_i\sigma_i$, $1/p = \sum_{i=1}^N \theta_i/p_i$, and $1/q = \sum_{i=1}^N \frac{\theta_i}{q_i}$. Then we have $\bigcap_{i=1}^N B^{\sigma_i}_{p_i,q_i} \subset B^0_{p,q}$ and

$$||v||_{B^\sigma_{p,q}} \leq C\prod_{i=1}^N ||v||^{\theta_i}_{B^{\sigma_i}_{p_iq_i}} \tag{2.15}$$

for all $v \in \bigcap_{i=1}^N B^{\sigma_i}_{p_iq_i}$

Lemma 2.8 Let $E(\cdot)$ be as in (2.3). let $1 < s < 2$. Then we have

$$||E(|u|^2)u||_{B^s_{4/3}} \leq C||u||_{B^s_4}||u||^2_{B^0_4} \tag{2.16}$$

$$||E(|u|^2)u||_{H^{2,4/3}} \leq C||u||_{H^{2,4}}||u||^2_{L^4} \tag{2.17}$$

Corollary 2.9 Let $n = 2,\ 3$. Let $E(\cdot)$ be as in (2.3). Let $1 < s < 2$. Then we have

$$||E(|u|^2)u||_{B^s_{4/3}} \leq C||u||^2_{H^1}||u||_{B^s_4} \tag{2.18}$$

$$||E(|u|^2)u||_{H^{2,4/3}} \leq C||u||^2_{H^1}||u||_{H^{2,4}} \tag{2.19}$$

Lemma 2.10. Let $\rho = 2n/(n-2+2\varepsilon), n \geq 2, \varepsilon \in (0,1)$.

(i) let $0 \leq \alpha < \alpha_1(n)$ and $\varepsilon = 1 - \frac{\alpha(n-2)}{4}$ if $n \geq 3$; $\varepsilon \in (0,1)$ is arbitrary if $n = 2$. we have

$$|||u|^\alpha u||_{H^{1,\rho'}} \leq C||u||^\alpha_{H^1}||u||_{H^{1,\rho}} \tag{2.20}$$

(ii) Let $1 < s \leq 2$, $1 \leq \alpha < \alpha_1(n)$, and $\varepsilon = 1 - \frac{\alpha(n-2)}{4}$ if $n \geq 3$; $\varepsilon \in (0,1)$ is arbitrary if $n = 2$. we have

$$|||u|^\alpha u||_{B^s_{\rho'}} \leq C||u||^\alpha_{H^1}||u||_{B^s_\rho} + C||u||^{\alpha-1}_{H^1}||u||_{H^{1,\rho}}||u||_{H^s} \tag{2.21}$$

(iii) Let $1 \leq s \leq 2$, $1 \leq \alpha_s(n)$, and $\varepsilon = 1 - \frac{\alpha(n-2s)}{4}$ if $s \leq \frac{n}{2}$; $\varepsilon = 1$ if $s > \frac{n}{2}$. We have

$$|||u|^\alpha u||_{B^s_{\rho'}} \leq C||u||^\alpha_{H^s}||u||_{B^s_\rho} \tag{2.22}$$

Lemma 2.11. (i) Let ρ be the same as in (iii)of Lemma 2.10, Let $1 \le s \le 2$.Then for any $\alpha \in (0, \alpha_s(n))$,we have

$$|||u|^\alpha u - |v|^\alpha v||_{L^{\rho'}} \le C(||u||_{H^s}^\alpha + ||v||_{H^s}^\alpha)||u-v||_{L^\rho} \tag{2.23}$$

(ii)We have

$$||E(|u|^2)u - E(|v|^2)v||_{L^{4/3}} \le C(||u||_{L^4}^2 + ||v||_{L^4}^2)||u-v||_{L^4}$$

3. Prof of Theorem 2.1

Let ρ be the same as in (iii) of Lemma 2.10. For the sake of convenience, we assume that $p_1 = 2$, $p_2 = \rho$, and $p_3 = 4$. Put

$$\mathcal{D} = \left\{ u \in \bigcap_{i=1}^{3} L^{\gamma(p_i)}(0,T; B_{p_i}^s) : ||u||_{\bigcap_{i=1}^3 L^{\gamma(p_i)}(0,T;B_{p_i}^s)} \le M \right\} \tag{2.24}$$

and for any $u, v \in \mathcal{D}$, we define a metric $d(u,v)$ by letting

$$d(u.v) = ||u-v||_{\bigcap_{i=1}^3 L^{\gamma(p_i)}(0,T;L^{p_i})} \tag{2.25}$$

Considering the mapping

$$J : u(t) \longrightarrow S(t)u_0 - i\int_0^t S(t-\tau)[a|u(\tau)|^\alpha - b_1 b_2 E(|u(\tau)|^2)]u(\tau)d\tau$$

we shall prove that J is a contraction mapping for some $T > 0$. For convenience, we denote $f_2(u) = a|u|^\alpha u$ and $f_3(u) = E(|u|^2)u$. For any $u, v \in D$, in view of (2.7)and (2.8) we have

$$||\mathcal{J}u||_{\bigcap_{i=1}^3 L^{\gamma(p_i)}(0,T;B_{p_i}^s)} \le C||u_0||_{H^s} + C\sum_{i=2}^{3} ||f_i(u)||_{L^{\gamma(p_i)'}(0,T;B_{p_i}^s)} \tag{2.26}$$

$$||\mathcal{J}u - \mathcal{J}v||_{\bigcap_{i=1}^3 L^{\gamma(p_i)}(0,T;L^{p_i})} \le C\sum_{i=2}^{3} ||f_i(u) - f_i(v)||_{L^{\gamma(p_i)'}(0,T;L^{p_i'})} \tag{2.27}$$

By Corollary 2.9 and Lemma 2.10, we have

$$\begin{aligned}||\mathcal{J}u||_{\bigcap_{i=1}^{3} L^{\gamma(p_i)}(0,T;B^s_{p_2})} &\leq C||u_0||_{H^s} + CT^{\delta_1}||u||^{\alpha}_{L^\infty(0,T;H^s)}||u||_{L^{\gamma(p_2)}(0,T;B^s_{p_2})} \\ &\quad + CT^{\delta_2}||u||^2_{L^\infty(0,T;H^1)}||u||_{L^{\gamma(p_3)}(0,T;B^s_{p_3})} \qquad (2.28)\end{aligned}$$

Where

$$\delta_1 = 1 - \frac{1}{\gamma(p_2)} = \epsilon \ , \quad \delta_2 = \begin{cases} \frac{1}{2}, & n = 2 \\ \frac{1}{4}, & n = 3 \end{cases}$$

ε is the same in (i) of Lemma 2.10. By Lemma 2.11, we have

$$\begin{aligned}||\mathcal{J}_u - \mathcal{J}_v||_{\cap_{i=1}^{3} L^{\gamma(p_i)}(0,T;L^{p_i})} &\leq CT^{\delta_1}(||u||^{\alpha} + ||v||^{\alpha})_{L^\infty(0,T;H^s)} \\ &\quad ||u - v||_{L^{\gamma(p_2)}(0,T;L^{p_2})} \\ &+ CT^{\delta_2}(||u||^2 + ||v||^2)_{L^\infty(0,T;H^1)} \\ &\quad ||u - v||_{L^{\gamma(p_3)}(0,T;L^{p_3})} \qquad (2.29)\end{aligned}$$

where $\delta_i (i = 1,2)$ are the same as the above .We have

$$||\mathcal{J}u||_{\cap_{i=1}^{3} L^{\gamma(p_i)}(0,T;B^s_{p_i})} \leq C||u_0||_{H^s} + CT^{\delta_1}M^{\alpha+1} + CT^{\delta_2}M^2 M \qquad (2.30)$$

$$||\mathcal{J}_u - \mathcal{J}_v||_{\cap_{i=1}^{3} L^{\gamma(p_i)}(0,T;L^{p_i})} \leq C(T^{\delta_1}M^{\alpha} + T^{\delta_2}M^2)||u - v||_{\cap_{i=1}^{3} L^{\gamma(p_i)}(0,T;L^{p_1})} \qquad (2.31)$$

Put $M = 2C||u_0||_{H^s}$. One can choose a sufficiently small $T > 0$ such that $C(T^{\delta_1}M^{\alpha}+T^{\delta_2}M^2) \leq \frac{1}{2}$. It follows from (2.30) and (2.31) that $\mathcal{J}$ is a contraction mapping on $(\mathcal{D}, d)$. Thus , $\mathcal{J}$ has a unique fixed point $u \in \mathcal{D}$ that is just the solution of the integral equation

$$u(t) = S(t)u_0 - i\int_0^t S(t-\tau)[a|u(\tau)|^{\alpha} - b_1 b_2 E(|u(\tau)|^{\tau})]u(\tau)d\tau \qquad (2.32)$$

Repeating the obove argument on $[T,T_1],[T_1,T_2],\cdots$, one can easily see that there exists a $T^*>0$ such that $u\in\bigcap_{i=1}^3 L^{\gamma(p_1)}(0,T;B^s_{p_i})$ is a unique solution of (2.32). Moreover, if $T^*<\infty$, by a standard disscussion, we have

$$\lim_{t\to T^*} sup||u(t)||_{H^s}=\infty \tag{2.33}$$

By virtue of (2.7) and (28), we have $u\in C_{loc}(0,T^*;H^s)\cap L^{\gamma(r)}_{loc}(0,T^*;B^s_r)$ for $r\in[2,2n/(n-2))$. This finishes the proof of Theorem 2.1

4 Proof of Theorem 2.2 and 2.3

Propasition 2.12 (Conservation law) Let u be a suitable smooth solution of (24). Then we have

$$||u(t)||_{L2}=||u_0||_{L^2},\quad \mathcal{E}(u(t))=\mathcal{E}(u_0) \tag{2.34}$$

Where

$$\mathcal{E}(u)=\frac{1}{2}||\nabla u||^2_{L^2}+\frac{a}{\alpha+2}||u||^{\alpha+2}_{L^{\alpha+2}}-\frac{b_1b_2}{4}||\left(\frac{\xi_1}{|\xi|}\right)\mathcal{F}|u|^2||^2_{L^2} \tag{2.35}$$

Lemma 2.13 Let $u_0\in H^s$ and $u\in H^s(s\geq 1)$ be a solution of (2.4). Assume that one of the following conditions holds:

$$\begin{array}{llll}
\text{(i)} & a>0 & \textit{and} & 2<\alpha<\infty;\\
\text{(ii)} & a>0,\ \ \alpha=2, & \textit{and} & a>b_1b_2;\\
\text{(iii)} & n=2,\ \ \alpha=2,\ \ b_1b_2\geq 0, & \textit{and} & (b_1b_2-a)||u_0||^2_{L^2}<4;\\
\text{(iv)} & n=2,\ \ \alpha=2,\ \ b_1b_2<0 & \textit{and} & -a||u_0||^2_{L^2}<4;\\
\text{(v)} & n=2,\ \ 0<\alpha<2, & \textit{and} & b_1b_2\int|u_0(x)|^2dx<4
\end{array}$$

Then we have $||u(t)||_{H^1}\leq C$, where C is independent of t.

Proof of Theorem 1.2 and 1.3. In view of Theorem 2.1, we shall show that $T^* = \infty$ by proving that $||u(t)||_{H^s}$ remains bounded on $(0, \infty)$. Let ρ be as in (ii) of lemma 2.10. It follows from proposition 2.4, Corollary 2.9 and (i) of Lemma 2.10 that for any $r \in [2, 2+\alpha_1(n))$,

$$\begin{aligned}
||u||_{L^{\gamma(r)}(0,T;H^{1,r})} &\leq C||u_0||_{H^1} + C|||u|^\alpha u||_{L^{\gamma(\rho)'}(0,T;H^{1,\rho'})} \\
&\quad + C||E(|u|^2)u||_{L^{\gamma(4)'}(0,T;H^{1,4/3})} \\
&\leq C||u_0||_{H^1} + CT^\epsilon ||u||^\alpha_{L^\infty(0,T;H^1)}||u||_{L^{\gamma(\rho)}(0,T;H^{1,\rho})} \\
&\quad + C||u_0||_{H^1} + CT^{1-n/4}||u||^2_{L_\infty(0,T;H^1)}||u||_{L^{\gamma(4)}(0,T;H^{1,4})}.
\end{aligned}$$

Since $||u(t)||_{H^1} \leq C_0$, where C_0 is independent of t, we can choose a sufficently small $T > 0$ such that

$$C(T^\epsilon C_0^\alpha + T^{1-n/4}C_0^2) \leq \frac{1}{2}$$

This leads to

$$||u||_{L^{\gamma(\rho)}(0,T;H^{1,\rho}) \cap L^{\gamma(4)}(0,T;H^{1,4})} \leq 2C||u_0||_{H^1} \leq 2CC_0 = C_1$$

Repeation the above procedure on $[T, 2T], [2T, 3T], \cdots$ we have

$$||u||_{L^{\gamma(\rho)}(nT,(n+1)T;H^{1,\rho}) \cap L^{\gamma(4)}(nT,(n+1)T;H^{1,4})} \leq 2CC_0$$

It follows that $u \in L^{\gamma(\rho)}_{loc}(0, \infty; H^{1,\rho}) \cap L^{\gamma(4)}_{loc}(0, \infty; H^{1,4})$. Moreover, one can easily see that $u \in L^{\gamma(r)}_{loc}(0, \infty; H^{1,r})$ for any $r \in [2, 2+\alpha_1(n)]$. For any $1 < s < 2$, in view of proposition 2.4, Corollary 2.9 and (ii) of Lemma 2.10, for any $r \in [2, 2+\alpha_1(n))$, we have

$$\begin{aligned}
||u||_{L^{\gamma(r)}(0,T;B^s_r)} &\leq C||u_0||_{H^s} + C|||u|^\alpha u||_{L^{\gamma(\rho)'}(0,T;B^s_{\rho'})} \\
&\quad + C||E(|u|^2)u||_{L^{\gamma(4)'}(0,T;B^s_{4/3})} \\
&\leq C||u_0||_{H^s} + CT^\epsilon ||u||^\gamma_{L^\infty(0,T;H^1)}||u||_{L^{\gamma(\rho)}(0,T;B^s_\rho)}
\end{aligned}$$

$$+ CT^{\varepsilon}||u||^{\alpha-1}_{L^\infty(0,T;H^1)}||u||_{L^{\gamma(\rho)}(0,T;H^{1,\rho})}||u||_{L^\infty(0,T;H^s)}$$

$$+ CT^{1-n/4}||u||^2_{L^\infty(0,T;H^1)}||u||_{L^{\gamma(4)}(0,T;B^s_4)}$$

$$\leq C||u_0||_{H^s} + CT^{\varepsilon}C_0^{\alpha}||u||_{L^{\gamma(\rho)}(0,T;B^s_\rho)}$$

$$+ CT^{\varepsilon}C_0^{\alpha-1}c_1||u||_{L^\infty(0,T;H^s)}$$

$$+ CT^{1-n/4}C_0^2||u||_{L^{\gamma(4)}(0,T;B^s_4)}.$$

Similarly as in the above process, we can choose a sufficiently small $T > 0$ such that

$$C(T^{1-n/4}C_0^2 + T^{\varepsilon}C_0^{\alpha-1}C_1 + T^{\varepsilon}C_0^{\alpha}) \leq \frac{1}{2}$$

This leads to

$$||u||_{L^\infty(0,T;H^s)\cap L^{\gamma(\rho)}(0,T;B^s_\rho)\cap L^{\gamma(4)}(0,T;B^s_4)} \leq 2 \subset ||u_0||_{H^s}$$

Repeating the above procedure, we obtain that $T^* = \infty$ in Theorem 2.1, *i.e.*, $u \in L^\infty_{loc}(0,\infty;H^s)\cap L^{\gamma(\rho)}_{loc}(0,\infty;B^s_\rho) \cap L^{\gamma(4)}_{loc}(0,\infty;B^s_4)$. It follows from proposition 2.4, $u \in L^{\gamma(r)}_{loc}(0,\infty;B^s_r)$ for any $r \in [2, 2+\alpha_1(n))$

For $s = 2$, in the same way as in th proof of the case $1 < s \subset 2$ we can prove that $u \in L^{\gamma(r)}_{loc}(0,\infty;H^{2,r})$for any $r \in [2, 2n/(n-2))$. The details are omitted .

Now we consider the following generalized Davey-Stewartson system

$$\begin{cases} iu_t + Au &= \lambda_1|u|^{p_1}u + \lambda_2|u|^{p^2}u + \mu u v_{x_1} \\ Bv &= (|u|^2)_{x_1} \end{cases} \tag{2.36}$$

Where $u(t,x)$ and $v(t,x)(x = x_1,\cdots,x_n))$ are complex and real valued functions of $(t,x) \in R^+ \times R^n$, respectively, $\lambda_1\lambda_2, \mu \in D$

$$A := \sum_{1\leq i,j\leq n} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} \quad , \quad B := \sum_{1\leq i,j\leq n} b_{ij}\frac{\partial^2}{\partial x_i \partial x_j}$$

(a_{ij}) and (b_{ij}) are all real and invertible matrices, in addition we assume that there exists a constant $C > 0$ satisfying

$$\left| \sum_{1 \le ij \le n} b_{ij}\xi_i\xi_j \right| \ge C|\xi|^2, \quad for \quad all \quad \xi \in R^n \tag{2.37}$$

We denote

$$E(\psi) = \mathcal{F}^{-1}\left[\frac{\xi_1^2}{\sum\limits_{1 \le i,j \le n} b_{ij}\xi_i\xi_j} \right] \mathcal{F}\psi \tag{2.38}$$

One find that the system (2.36) can be rewritten as

$$iu_t + Au = \lambda_1|u|^{p_1}u + \lambda_2|u|^{p_2}u + \mu E(|u|^2)u \tag{2.39}$$

For any $4/n \le p < \infty$ and $r \in [2, \infty)$ and we write

$$S(p) = \frac{n}{2} - \frac{2}{p} \ , \quad \frac{2}{\gamma(r)} = n(\frac{1}{2} - \frac{1}{r}) \ , \quad r(p) = \frac{2n(2+p)}{n(2+p) - 4} \tag{2.40}$$

Let

$$\alpha(n) = \begin{cases} \infty & n = 2 \\ 2n/(n-2) & n > 2 \end{cases} \tag{2.41}$$

For the equation(3.39), $p = 4/(n-2s)$ is said to be an H^s –critical power and $p < 4/(n-2s)$ is called an H^s–subcritical power . It is easy to see that every $p \ge \frac{4}{n}$ is just an $H^{s(p)}$–critical power. In the sequel, we always assume that (a_{ij}) and (b_{ij}) are invertible and (b_{ij}) satisfies (2.37). For any $r \in [1, \infty], r'$ denotes the dual number of $r, i.e., \frac{1}{r} + \frac{1}{r'} = 1$. Our main results are the following:

Theorem 2.14. let $n \ge 2$, $4/n \le p_1 \le p_2 < \infty$, $max(s(2), s(p_2) \le s < \infty$ and $[s] \le p_1$. let $u_0 \in H^s$. Then there exists a $T^* > 0$ such that (2.39) with the initial value u_0 at $t = 0$ has a unique solution $u \in C_{loc}(o, T^*; H^s) \cap L^{\gamma(r)}_{loc}(0, T^*; B^s_{r,2})$ for all $r \in [2, \alpha(n))$, Moreover, if

$T^*; < \infty$, then

$$||u||_{\cap_{p=2,p_1p_2} L^{2+p}(0,T^*;B^{s(p)}_{r(p),2})} = \infty \tag{2.42}$$

For the sake of convenivence,we write

$$B^{s_0,s}_{\delta} = \{u \in H^s : ||u||_{\dot{H}^{s_0}} \leq \delta\} \tag{2.43}$$

For any $0 \leq s_0 < s < \infty$.

Theorem 2.15 Let $n \geq 2$, $4/n \leq p_1 \leq p_2 < \infty$, $max(s(2),\ s(p)_2 \leq s < \infty$ and $[s] \leq p_1$ There exists a $\delta > 0$ such that if $u_0 \in \cap_{p=2,\ p_1,\ p_2} B^{s(p),s}_{\delta}$, then(2.39) with the initial value u_0 at $t = 0$ has a unique solution $u \in C(0,\infty; H^s) \cap L^{\gamma(r)}(0,\infty; B^s_{r,2})$ for all $r \in [2, \alpha(n))$.

Theorem 2.16 Let $S(t)$ be the unitary group generated by $i\frac{\partial}{\partial e} + A$. let $n \geq 2$, $4/n \leq p_1 \leq p_2 < \infty$, $max[s(2), s(p_2)) \leq s < \infty$. and $[s] \leq p_1$. There exists a $\delta > 0$ such that the scattering operator S of (2.39) map $\bigcap_{p=2,\ p_1,\ p_2)} B^{s(p),s}_{\delta}$ into H^s. More precisely, for any $\bar{\varphi} \in \bigcap_{p=2,p_1,p_2} B^{s(p),s}_{\delta}$,(2,39) has a unique solution $u \in C(R; H^s) \cap L^{\gamma(r)}(R; B^s_{r,2})$ for all $r \in [2, \alpha(n))$ such that

$$||u(t) - S(t)\varphi^-||_{H^s} \longrightarrow 0, \quad as\ t \longrightarrow -\infty;$$

and there exists $\varphi^+ \in H^s$ such that the above solution u satistying

$$||u(t) - S(t)\varphi^+||_{H^s} \longrightarrow 0, \quad as\ t \longrightarrow \infty$$

Remark 2.16 Since (a_{ij}) is only assumed to be invertible. A can be a hyperbolic operator in Theorem 2.14-2.16. for example, $A = \sum_{i\in N_1} \frac{\partial^2}{\partial x_i^2} - \sum_{j\in N_2} \frac{\partial^2}{\partial x_j^2}$, $N_1 \cup N_2 = \{1, \cdots, n\}$ In view of condition (2.37) the operator B is essentially elliptical.

Remark 2.17. $[s] \leq p_1$ is used for deriving the differentiability of $|u|^{p_1}u$, so, if p_i are all even integers $C_i = 1, 2$, then condition $[s] \leq p_1$ could be removed in Theorem 2.14–2.16 .

Remark 2.18 . In Theorem 2.14, if $T^* < \infty$, then the solution u actually blow up in the Besov space $B^{s(2\mathrm{v}p_2)}_{r(2\mathrm{v}p_2)}$, where $s(2\mathrm{v}p_2) = smax(2, p_2)$ is the critical order association with the nonlinearity $|u|^{2\mathrm{v}p_2}$ u, i. e.,

$$||u||_{L^{2+(2\mathrm{v}p_2)}}(0, T^*; B^{s(2\mathrm{v}p_2)}_{r(2\mathrm{v}p_2),2}) = \infty \tag{2.44}$$

It means that $\bigcap_{p=2,p_1p_2} L^{\gamma(p)}(0, T^*; B^{s(p)}_{r(p_2)} \supset L^{r(2\mathrm{v}p_2)}(0, T^*; B^{s(2\mathrm{v}p_2)}_{r(2\mathrm{v}p_2),2})$, whence , (2.44) follows .

Remark 2.19. Considering an important case $p_1 = p_2 = 2$ in Theorem 2.15 ,we have shown that (2.39) with the initial value u_0 at $t = 0$ has a unique solution $u \in C(0, \infty; H^s) \cap L^4(0, \infty; B^s_{r(2),2})$ if $||u_0||_{\dot{H}^{\frac{n}{2}-1}} \leq \delta, s \geq n/2 - 1$.

Remark 2.20. one see that

$$\cap_{p=2,\ p_1,\ p_2} B^{s(p),s}_s = \{u \in H^s : ||u||_{\cap_{p=2,\ p_1,\ p_2} \dot{H}^{s(p)} \leq \delta}\}$$

in Theorem 2.15, $||u_0||_{H^s}$ can be arbitrarily large if $s > (n/2 - 1)\mathrm{v}s(p_2)$.

3 Existence of global attractor for Davey-Stewartson systems

First, we consider the following Davey-Stewartson systems

$$\begin{cases} i\frac{\partial u}{\partial e} + \Delta u + i\delta u = \alpha|u|^2 u + bu\frac{\partial \varphi}{\partial x} + f(x, y), & (3.1) \\ \Delta\varphi = \frac{\partial}{\partial x}(|u|^2), & (3.2) \end{cases}$$

where $f(x, y) \in L^2(R^2), \delta > 0$ and

$$\alpha \leq 0 \quad , \quad \alpha + b \leq 0 \tag{3.3}$$

Obviously, systems (3.1) (3.2) can be reduced to a nonlinear nonlocal Schrodinger equations

$$i\frac{\partial u}{\partial t}+\Delta u+i\delta u=\alpha|u|^2u+buE(|u|^2)+f(x,y)\ , \tag{3.4}$$

which is complemented with the initial condition

$$u(x,y,o)=u_0(x,y)$$

where

$$\hat{E}(f)(\xi_1\xi_2)\frac{\xi_1^2}{\xi_1^2+\xi_2^2}\hat{f}(\xi_1,\xi_2), \tag{3.5}$$

Theorem 3.1 Assnme that (3.3) holds. Then there exists a compact global attracfor for systems(3.4)(3.5)

Second, we consider the following Darey-Stewartson system

$$\begin{cases} \frac{\partial A}{\partial t}-a\frac{\partial^2 A}{\partial x^2}-b\frac{\partial^2 A}{\partial y^2}=XA-\beta|A|^2A+\gamma QA & (3.6)\\ \frac{\partial^2 Q}{\partial x^2}+\frac{\partial^2 Q}{\partial y^2}=\frac{\partial^2}{\partial y^2}(|A|^2) \qquad t>0,\ (x,y)\in\Omega, & (3.7)\end{cases}$$

supplemented with boundary conditions

$$A(t,x,y)=0\ \ ,\ \ \alpha(t,x,y)=0\ \ ,\ \ t\geq 0\ \ ,\ \ (x,y)\in\Omega \tag{3.8}$$

and initial condition

$$A(0,x,y)=A_0(x,y)\ \ ,\ \ (x,y)\in\Omega\ , \tag{3.9}$$

where $a=a_1+ia_2$, $b=b_1+ib_2$, $\beta=\beta_1+i\beta_2$, $\gamma=\gamma_1+i\gamma_2$ and $\chi=\chi_1+i\chi_2$ are complex constants , $\Omega\subset R^2$ is a smooth bunded domain. We can reduce (3.6) (3.7) to a nonlocal nonlinear Schrödinger equation

$$\begin{cases} \frac{\partial A}{\partial e}-a\frac{\partial^2 A}{\partial x^2}-b\frac{\partial^2 A}{\partial y^2}=\chi A-\beta|A|^2A-\gamma AE(|A|^2),\ \ t>0\ \ (x,y)\in\Omega & (3.10)\\ A(t,x,y)=0,\ \ t\geq 0,; \qquad (x,y)\in\partial\Omega & (3.11)\\ A(0,x,y)=A_0(x,y), \qquad (x,y)\in\Omega & (3.12)\end{cases}$$

where $E(|A|^2) = -(-\triangle)^{-1}\frac{\partial^2}{\partial y^2}|A|^2$

Theorem 3.2 Assume that

[H] $\quad K = min\{a_1, b_1\} > 0, \beta_1 > 0, \beta_1 + c(2)\gamma_1 > 0, X_1 > 0$

holds, $C(2)$ is a miniml constant, such that

$$||\frac{\partial^2 u}{\partial y^2}||_2 \leq c(2)||\Delta u||_2 \ , \ u \in c_0^\infty(\Omega)$$

Then there exists a global compact attractor for system (3.10)-(3.12), which has finite dimensional Hausdorff dimension and fractal dinension

4 Approximate inertial manifolds

We consider the approximate inertial manifolds. for systems (3.10)-(3.12), we have

Theorem 4.1 Assume that [H] holds, $u_0 \in L^p(\Omega)$ $(p > 3)$, $||u_0||_p \leq R$. Then there exists the flat approximate inertial manifold M_0 and non flat approximate manifold M_1 for system (3.10)-(3.12). *i.e* ., the orbits of systen(3.10)-(3.12) from u_0 when $t > T_* > 0$ remuin at a distance H of M_0 and M_1 bounded by $Ke^{-\sigma\delta}$. $\sigma\delta > 0, K > 0$.

5 Existence and Blow Up of Solution to a Degenerate D S Equation

We study the following degenerate Davey-Stewartson equations

$$i\psi_t + \psi_{xx} = \chi\psi \tag{5.1}$$

$$\chi_y = |\psi|_x^2 \tag{5.2}$$

With initial condition

$$\psi(0,x,y) = \psi_0(x,y), \qquad (x,y) \in R^2 \tag{5.3}$$

At infinity we assume that

$$\lim_{|x|,|y|\to\infty} \psi(t,x,y) = 0, \qquad \lim_{|x|,|y|\to\infty} \chi(t,x,y) = 0 \tag{5.4}$$

We have

Theorem 5.1. If $\psi_0 \in L^2(R^2)$ with $\psi_{0x} \in L^2(R^2)$satifying

$$\int_{R^2} |\psi_0|^2 dxdy < \frac{1}{2}$$

then(5.17)-(5.4) has a global weak solution,*i.e.*

$$\psi, \psi_x \in L^{R^\infty}(L^+(L^2(R^2))$$

$$\chi \in L^\infty(R^+; L^2_{loc}(R^2), \chi_y \in L^\infty(R^+; L^1(R^2)$$

and if they satify(5.1)in the sense of $L^\infty(R^+; H^{-1}(R^2)$, and (5.2)in the sense of $L^\infty(R^+; L^1(R^2))$.

Theorem 5.2 Let $\psi_0 \in L^2(R^2)$ with $x\psi_0 \in L^2(R^2), \psi$ be the solution of (5.1), (5.2)with $x\psi \in L^2(R)$. If one of the following conditions holds ,

(i) $E(0) = \int_{R^2} |\psi_x|^2 dxdy + \frac{1}{2}\int_{R^2} \chi|\psi|^2 dxdy < 0$

(ii) $E(0) = 0$ *and* $Im \int_{R^2} x\psi_0\overline{\psi}_{0x} dxdy > 0$

(iii) $E(0) > 0$ *and* $Im \int_{R^2} x\psi_0\overline{\psi}_{0x} dxdy > 4\sqrt{E(0)I(0)}$

$I(0) = \int_{R^2} x^2|\psi_0|^2 dxdy$

then

$$\lim_{t\to T^*} \inf ||\psi_x||_2^2 = \infty$$

that is, the solution will blow up in finite time.

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